Modified Bernstein Polynomials and Jacobi Polynomials in q-Calculus

Marie-Madeleine Derriennic

IRMAR, INSA, 20 Avenue des Buttes de Coësmes, CS 14315,35043-RENNES, FRANCE

E-mail adress: mderrien@insa-rennes.fr

Abstract

We introduce here a generalization of the modified Bernstein polynomials for Jacobi weights using the q-Bernstein basis proposed by G.M. Phillips to generalize classical Bernstein Polynomials. The function is evaluated at points which are in geometric progression in $]0,1[$. Numerous properties of the modified Bernstein Polynomials are extended to their q-analogues: simultaneous approximation, pointwise convergence even for unbounded functions, shape-preserving property, Voronovskaya theorem, self-adjointness. Some properties of the eigenvectors, which are q-extensions of Jacobi polynomials, are given.

Keywords: q-Bernstein, q-Jacobi, Bernstein-Durrmeyer, totally positive, simultaneous approximation.

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1 Introduction

G.M.Phillips has proposed a generalization of Bernstein polynomials based on the q-integers (cf. [9]). We introduce here a q-analogue of the operators which are often called Bernstein Durrmeyer polynomials and denoted $M_{n,1}^{\alpha,\beta}$ (cf. [3],[2]).

In all the paper, we shall assume that $q \in]0,1[$ and, $\alpha, \beta > -1$ (part 5 excepted).

For any integer n, and a function f defined on $]0,1[$ we set

$$M_{n,q}^{\alpha,\beta} f(x) = \sum_{k=0}^{n} f_{n,k,q}^{\alpha,\beta}\; b_{n,k,q}(x) \tag{1}$$

where each $f_{n,k,q}^{\alpha,\beta}$ is a mean of f defined by Jackson integrals. The polynomials $b_{n,k,q}(x)$ are q-analogues of the Bernstein basis polynomials and are defined by $b_{n,k,q}(x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (1-x)_q^{n-k}$, with $\begin{bmatrix} n \\ k \end{bmatrix}_q = \dfrac{[n]_q!}{[k]_q!\,[n-k]_q!}$, ($q$-binomial coefficient), $k = 0, \ldots, n$. They verify $\sum_{k=0}^{n} b_{n,k,q}(x) = 1$ (cf. [9]).

We follow the definitions and notations of ([7]).

For any real a, $[a]_q = (1-q^{a)}/(1-q)$, $\Gamma_q(a+1) = (1-q)_q^a/(1-q)^a$,

(if $a \in \mathbb{N}$ the q-integer $[a]_q$ is $[a]_q = 1 + q + \cdots + q^{a-1}$ and $\Gamma_q(a+1) = ([a]_q)!$);

$(1-x)_q^a = \prod_{j=0}^{\infty}(1-q^j x) \Big/ \prod_{j=0}^{\infty}(1-q^{j+a}x)$ and consequently $(1-x)_q^{a+b} = (1-x)_q^a (1-q^a x)_q^b$ holds for any b, $(1-x)_q^m = \prod_{j=0}^{m-1}(1-q^j x)$ if m is integer.

The notations will be simplified as much as possible, the superscript α, β and the index q when q is fixed, will be suppressed in some proofs.

We introduce the positive bilinear form:

$$\langle f, g \rangle_q^{\alpha,\beta} = q^{(\alpha+1)(\beta+1)}(1-q) \sum_{i=0}^{\infty} q^i q^{i\alpha} (1-q^{i+1})_q^{\beta} f(q^{i+\beta+1}) g(q^{i+\beta+1}), \tag{2}$$

whenever it is defined. It can be written under the form of two definite q-integrals

$$\begin{aligned} \langle f,g\rangle_q^{\alpha,\beta} &= \int_0^{q^{\beta+1}} t^\alpha(1-q^{-\beta}t)_q^\beta f(t)g(t)d_qt \\ \text{and } \langle f,g\rangle_q^{\alpha,\beta} &= q^{(\alpha+1)(\beta+1)}\int_0^1 t^\alpha(1-qt)_q^\beta f(q^{\beta+1}t)g(q^{\beta+1}t)d_qt. \end{aligned}$$

(the definite q-integral of a function f is $\int_0^a f(x)d_qx = a(1-q)\sum_{i=0}^\infty q^i f(q^i a)$ (cf. [7]))

Definition 1 *We set in formula (1) :*

$$f_{n,k,q}^{\alpha,\beta} = \frac{\langle b_{n,k,q}, f\rangle_q^{\alpha,\beta}}{\langle b_{n,k,q}, 1\rangle_q^{\alpha,\beta}} = \frac{\int_0^1 t^{k+\alpha}\ (1-qt)_q^{n-k+\beta}\ f(q^{\beta+1}t)\ d_qt}{\int_0^1 t^{k+\alpha}\ (1-qt)_q^{n-k+\beta}\ d_qt},\ \ k=0,...,n, \tag{3}$$

$$\textit{to define}\quad M_{n,q}^{\alpha,\beta}f(x) = \sum_{k=0}^n \frac{\langle b_{n,k,q}, f\rangle_q^{\alpha,\beta}}{\langle b_{n,k,q}, 1\rangle_q^{\alpha,\beta}} b_{n,k,q}(x). \tag{4}$$

We see that the polynomial $M_{n,q}^{\alpha,\beta}f$ is well defined if there exists $\gamma \geq 0$ such that $x^\gamma f(x)$ is bounded on $]0,A]$ for some $A \in]0,1]$ and $\alpha > \gamma - 1$. Indeed, $x^\alpha f(x)$ is then q-integrable for the weight $w_q^{\alpha,\beta}(x) = x^\alpha(1-qx)_q^\beta$. We will say, in this case, that f satisfies the condition $C(\alpha)$. Also $\langle f,g\rangle_q^{\alpha,\beta}$ is well defined if the product fg satisfies $C(\alpha)$, particularly if f^2 and g^2 do it.

In many cases, the limit of $M_{n,q}^{\alpha,\beta}f(x)$ when q tends to 1 is :

$$M_{n,1}^{\alpha,\beta}f(x) = \sum_{k=0}^n \left(\int_0^1 t^{k+\alpha}\ (1-t)^{n-k+\beta}\ f(t)\ dt \Big/ \int_0^1 t^{k+\alpha}\ (1-t)^{n-k+\beta}\ dt\right) b_{n,k}(x)$$

with $b_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k}$.

Numerous properties of the operator $M_{n,1}^{\alpha,\beta}$ will be extended to $M_{n,q}^{\alpha,\beta}$ in this paper.

2 First properties

For any $n \in \mathbb{N}$, the operator $M_{n,q}^{\alpha,\beta}$ has the following properties.

- It is linear, positive and it preserves the constants so it is a contraction $\left(\sup_{x\in[0,1]} \left|M_{n,q}^{\alpha,\beta} f(x)\right| \leq \sup_{x\in]0,1[} |f(x)|\right)$.

- It is self-adjoint: $\langle M_{n,q}^{\alpha,\beta} f, g\rangle_q^{\alpha,\beta} = \langle f, M_{n,q}^{\alpha,\beta} g\rangle_q^{\alpha,\beta}$.

- It preserves the degrees of the polynomials of degree $\leq n$.

The first properties are consequences of the definition. The last one follows after the following proposition since $D_q x^p = [p]\, x^{p-1}$.

Proposition 1 *If f verifies the condition $C(\alpha)$, we have :*

$$D_q M_{n,q}^{\alpha,\beta} f(x) = \frac{[n]_q}{[n+\alpha+\beta+2]_q} q^{\alpha+\beta+2} M_{n-1,q}^{\alpha+1,\beta+1}\left(D_q f\left(\frac{\cdot}{q}\right)\right)(qx), x \in [0,1], \tag{5}$$

where the q-derivative of a function f is $D_q f(x) = \dfrac{f(qx) - f(x)}{(q-1)x}$ if $x \neq 0$.

(When f' is continuous on $[0,1]$, the limit of formula (5) is, when q tends to 1, $\left(M_{n,1}^{\alpha,\beta} f\right)'(x) = n\,(n+\alpha+\beta+2)^{-1}\, M_{n-1,1}^{\alpha+1,\beta+1}\,(f')\,(x))$ (cf. [4])).

Proof. We compute $Db_{n,k}(x) = [n]\,(b_{n-1,k-1}(qx)/q^{k-1} - b_{n-1,k}(qx)/q^k)$ if $1 \leq k \leq n-1$ and $Db_{n,0}(x) = -[n]\,b_{n-1,0}(qx)$, $Db_{n,n}(x) = [n]\,b_{n-1,n-1}(qx)/q^{n-1}$ to get $DM_n^{\alpha,\beta} f(x) = [n] \sum_{k=0}^{n-1} b_{n-1,k}(qx)(f_{n,k+1}^{\alpha,\beta} - f_{n,k}^{\alpha,\beta})/q^k$.

We denote $\psi_{n,k}^{\alpha,\beta}(t) = t^{k+\alpha}(1-qt)_q^{n-k+\beta}$, $k = 0,\ldots,n$.

Recall that the q-derivative of $g_1 g_2$ is $D_q(g_1 g_2)(x) = D_q g_1(x) g_2(qx) + g_1(x) D_q g_2(x)$. The q-Beta functions are $B_q(u,v) = \int_0^1 t^{u-1}(1-qt)_q^{v-1} d_q t = \Gamma_q(u)\Gamma_q(v)/\Gamma_q(u+v)$. The function $\psi_{n-1,k}^{\alpha+1,\beta+1}(\frac{t}{q}) f(q^{\beta+1}t)$, $t \in]0,1[$, extended by 0 in 0 is continuous at 0.

Hence we may use a q-integration by parts to write, for $k = 0, \ldots, n-1$:

$B_q(k+\alpha+2, n-k+\beta+1)\,[n+\alpha+\beta+2]\,(f^{\alpha,\beta}_{n,k+1} - f^{\alpha,\beta}_{n,k}) =$

$$-q^{k+\alpha}\int_0^1 (D\psi^{\alpha+1,\beta+1}_{n-1,k})(\tfrac{t}{q})f(q^{\beta+1}t)d_qt = \int_0^1 q^{k+\alpha+\beta+2}\psi^{\alpha+1,\beta+1}_{n-1,k}(t)(Df)(q^{\beta+1}t)d_qt$$

$- \left[\psi^{\alpha+1,\beta+1}_{n-1,k}(\tfrac{t}{q})f(q^{\beta+1}t)\right]_0^1.$

Hence $(f^{\alpha,\beta}_{n,k+1} - f^{\alpha,\beta}_{n,k}) = \frac{q^{\alpha+\beta+2+k}}{[n+\alpha+\beta+2]}\frac{\int_0^1 t^{k+\alpha+1}(1-qt)^{n-k+\beta}(Df)(q^{\beta+1}t)d_qt}{B_q(k+\alpha+2,n-k+\beta+1)}$ and

$$DM^{\alpha,\beta}_n f(x) = \frac{[n]\,q^{\alpha+\beta+2}}{[n+\alpha+\beta+2]}\sum_{k=0}^{n-1}\frac{\langle b_{n-1,k}, Df(\frac{\cdot}{q})\rangle^{\alpha+1,\beta+1}_q}{B_q(k+\alpha+2,n-k+\beta+1)}b_{n-1,k}(x). \quad \blacksquare$$

Theorem 1 *The following equality holds for any $x \in [0,1]$:*

$$M^{\alpha,\beta}_{n,q}f(x) = \sum_{j=0}^{\infty}\Phi^{\alpha,\beta}_{j,n,q}(x)f(q^{j+\beta+1}) \tag{6}$$

$$\begin{aligned} \textit{where, }\ \Phi^{\alpha,\beta}_{j,n,q}(x) &= u_j\sum_{k=0}^{n} v_k b_{n,k,q}(q^{j+\beta+1})b_{n,k,q}(x), \\ u_j &= (1-q)q^{j(\alpha+1)}(1-q^{j+1})^{\beta}_q, \quad j \in \mathbb{N}, \\ v_k^{-1} &= q^{k(\beta+1)}\begin{bmatrix} n \\ k \end{bmatrix}_q B_q(k+\alpha+1, n-k+\beta+1), \quad k = 0, \ldots, n. \end{aligned}$$

Moreover, for any $r \in \mathbb{N}$, the sequence $\Phi^{\alpha,\beta}_{r,n,q}, \Phi^{\alpha,\beta}_{r-1,n,q}, \ldots, \Phi^{\alpha,\beta}_{0,n,q}$ is totally positive, that is to say, the collocation matrix $\left(\Phi^{\alpha,\beta}_{r-j,n,q}(x_i)\right)_{i=1,\ldots,m,j=0,\ldots,r}$ is totally positive for any family (x_i), $0 \le x_1 < \ldots < x_m \le 1$.

Proof. We set $\Phi_j = \Phi^{\alpha,\beta}_{j,q}, b_{n,k,q} = b_k, c = q^{\beta+1}$. The formulae (6) come by writing the definite q-integrals $\langle b_k, f\rangle^{\alpha,\beta}_q$ as discrete sums in (4) and the Beta integrals $\langle b_k, 1\rangle^{\alpha,\beta}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q B_q(k+\alpha+1, n-k+\beta+1)$, $k = 0, \ldots, n$.

For the total positivity of the Φ_j, we have to prove that, for any $m \in \mathbb{N}$ and any two families $(x_i)_{i=1,\ldots,m}, (j_k)_{k=1,\ldots,m}$, with $0 \le x_1 \le \ldots \le x_m \le 1,\ 0 \le j_m \le \ldots \le j_1$, the determinant $\det(\Phi_{j_l}(x_i))_{i=1,\ldots,m,l=1,\ldots,m}$ is non negative. From the multilinearity of

the determinants, there is a basic composition formula for the discrete sums (cf. [8]).

We have $\det(\Phi_{j_l}(x_i))_{i=1,\ldots,m,l=1,\ldots,m} = \prod_{l=1}^m u_{j_l} E$ where

$$
\begin{aligned}
E &= \det\left(\sum_{k=0}^n v_k b_k(cq^{j_l}) b_k(x_i)\right)_{i=1,\ldots,m,l=1,\ldots,m} \\
&= \sum_{k_1=0}^n \ldots \sum_{k_m=0}^n v_{k_1} \ldots v_{k_m} \det(b_{k_i}(cq^{j_l}) b_{k_i}(x_i))_{i=1,\ldots,m,l=1,\ldots,m} \\
&= \sum_{k_1=0}^n \ldots \sum_{k_m=0}^n v_{k_1} \ldots v_{k_m} b_{k_1}(x_1) \ldots b_{k_m}(x_m) \det(b_{k_i}(cq^{j_l}))_{i=1,\ldots,m,l=1,\ldots,m} \\
&= \sum_{0 \le k_1 \le \ldots \le k_m \le n} v_{k_1} \ldots v_{k_m} \det(b_{k_l}(x_i))_{i=1,\ldots,m,l=1,\ldots,m} \det(b_{k_i}(cq^{j_l}))_{i=1,\ldots,m,l=1,\ldots,m}.
\end{aligned}
$$

We know that the q-Bernstein basis is totally positive (cf. [6]). Hence we have $\det(b_{k_l}(x_i))_{i=1,\ldots,m,l=1,\ldots,m} \geq 0$ and also $\det(b_{k_i}(cq^{j_l}))_{i=1,\ldots,m,l=1,\ldots,m} \geq 0$, since $cq^{j_1} < cq^{j_2} < \ldots < cq^{j_m}$. So E is non negative and the result follows. ■

Corollary 1 *The number of sign changes of the polynomial $M_{n,q}^{\alpha,\beta} f$ on $]0,1[$ is not greater than the number of sign changes of the function f.*

Proof. For any $r \in \mathbb{N}$, the sequence $\Phi_r, \Phi_{r-1}, \ldots, \Phi_0$ is totally positive. We deduce that the number of sign changes of the polynomial $\sum_{j=0}^r \Phi_j(x) f(q^{j+\beta+1})$ is not greater than the number of sign changes of the sequence $f(q^{j+\beta+1})$, $j = 0, \ldots, r$, hence not greater than the number of sign changes of the function f in $]0,1[$ (cf. [5]). When r tends to infinity this property is preserved hence is true for $M_n f$. ■

Corollary 2 *Let f be a function satisfying the condition $C(\alpha)$.*

1. *If f is increasing (respectively decreasing), then the function $M_{n,q}^{\alpha,\beta} f$ is increasing (respectively decreasing).*

2. *If f is convex, then the function $M_{n,q}^{\alpha,\beta} f$ is convex.*

Proof. **1)** If f is monotone, for any $s \in \mathbb{R}$ the function $f-s$ has at most one sign change. Hence $M_n^{\alpha,\beta}(f-s) = M_n^{\alpha,\beta} f - s$ has at most one sign change and $M_n^{\alpha,\beta} f$ is monotone. If f is increasing, $Df(\frac{\cdot}{q})$ is positive on $]0,q[$. Since the operators $M_n^{\alpha,\beta}$ are positive, we obtain for $x \in [0,1]$, $M_{n-1}^{\alpha+1,\beta+1}\left(Df\left(\frac{\cdot}{q}\right)\right)(qx) \geq 0$, and, using (5), $DM_n^{\alpha,\beta} f(x) \geq 0$. So the function $M_n^{\alpha,\beta} f$ is increasing.

2) Let suppose the function f is convex. Since $M_n^{\alpha,\beta}$ preserves the degree of the polynomials, for any real numbers γ_1, γ_2 there exist δ_1, δ_2 and a function g such that $g(x) = f(x) - \delta_1 x - \delta_2$ and $M_n^{\alpha,\beta} f(x) - \gamma_1 x - \gamma_2 = M_n^{\alpha,\beta} g(x)$. The number of sign changes of g being at most two, it is the same for $M_n^{\alpha,\beta} g$. Hence $M_n^{\alpha,\beta} f$ is convex or concave. Moreover, if a function φ is convex (respectively concave), $D^2\varphi(x) = \frac{q^3}{(q-1)^2 x^2}(\varphi(q^2 x) - [2]\,\varphi(qx) + q\varphi(x))$ is ≥ 0 (respectively ≤ 0). Hence $M_{n-2}^{\alpha+2,\beta+2}\left(D^2\varphi\left(\frac{\cdot}{q}\right)\right)(q^2 x) \geq 0$. Using (5) two times we obtain $D^2 M_n^{\alpha,\beta}\varphi(x) \geq 0$ and $M_n^{\alpha,\beta}\varphi$ is not concave. ∎

3 Convergence properties

Theorem 2 *If f is continuous on $[0,1]$,*

$$\left\| M_{n,q}^{\alpha,\beta} f - f \right\|_\infty \leq C_{\alpha,\beta}\ \omega\left(f, \frac{1}{\sqrt{[n]_q}}\right),$$

where $\|f\|_\infty$ is the uniform norm of f on $[0,1]$ and $\omega(f,.)$ is the usual modulus of continuity of f, the constant $C_{\alpha,\beta}$ being independent of n, q, f.

Proof. As $M_n^{\alpha,\beta}$ is positive, O. Shisha and B. Mond theorem can be applied. It is sufficient to prove that the order of approximation of f by $M_n^{\alpha,\beta} f$ is $O(\frac{1}{[n]})$ for the

functions $f_i(x) = x^i$, $i = 0, 1, 2$. We compute the polynomials $M_n^{\alpha,\beta} f_i$, $i = 1$ and 2, with the help of (5) by q-integrations.

$[n + \alpha + \beta + 2] \, (M_n^{\alpha,\beta} f_1(x) - x) = q^{\beta+1} \, [\alpha + 1] - x \, [\alpha + \beta + 2]$ and

$[n + \alpha + \beta + 2] \, [n + \alpha + \beta + 3] \, (M_n^{\alpha,\beta} f_2(x) - x^2) =$

$[n] \, [2] \, x(q^{\alpha+2\beta+3} \, [\alpha{+}2] \, (1 - x) - [\beta{+}1] \, x) + [a{+}\beta{+}3] \, [\alpha{+}\beta{+}2] \, x^2 + q^{2\beta+2} \, [\alpha{+}2] \, [\alpha{+}1] \, .$

The result follows since $0 < q < 1$ and $0 \leq [a] \leq \max(a, 1)$ if $a \geq 0$. ■

Remark 1

In order to have uniform convergence for all continuous functions on $[0, 1]$, it is sufficient to have $\lim\limits_{n\to\infty} M_{n,q}^{\alpha,\beta} f_i = f_i$ for $i = 1, 2$, hence $\lim\limits_{n\to\infty} 1/\, [n]_q = 0$. This is realized if and only if $q = q_n$ and $\lim\limits_{n\to\infty} q_n = 1$. Indeed, for every $n \in \mathbb{N}$, in both cases $q^n < 1/2$ and $q^n \geq 1/2$, we have $1 - q < 1/\, [n]_q \leq 2 \max(1 - q, \ln 2/n)$. To maximize the order of approximation by the operator $M_{n,q_n}^{\alpha,\beta}$, we are interested to have $[n]_{q_n}$ of the same order as n, that is to say to have $\rho n < [n]_{q_n} \leq n$, for some $\rho > 0$, property which holds with the following property S for (q_n).

Definition 2 *The sequence $(q_n)_{n\in\mathbb{N}}$, has the property S if and only if there exists $N \in \mathbb{N}$ and $c > 0$ such that, for any $n > N$, $1 - q_n < c/n$.*

Lemma 1

The property S holds if and only if the property P_1 (respectively P_2) holds where :

P_1 is "There exists $N_1 \in \mathbb{N}$ and $c_1 > 0$ such that, for any $n > N_1$, $[n]_{q_n} \geq c_1 n$",

P_2 is "There exists $N_2 \in \mathbb{N}$ and $c_2 > 0$ such that, for any $n > N_2$, $q_n^n \geq c_2$".

Proof. For any $n \in \mathbb{N}$, the function $\xi(x) = (1 - x^n)/(1 - x)$ is increasing on $[0, 1[$. If S holds, we have, for any $n > N_1 = N$, $[n]_{q_n} = \xi(q_n) \geq \xi(1 - c/n) \geq n(1 - e^{-c})/c$

and P_1 follows. If P_1 holds, we have, for any $n > N = N_1$, $1/(1-q_n) \geq [n]_{q_n} \geq c_1 n$ and S follows. If P_2 holds, we have, for any $n > N = N_2$, $n(1-q_n) \leq -n \ln q_n < -\ln c_2$ and S follows. If S holds, there exists $N_2 > N$ such that, if $n > N_2$, $1 - q_n < 1/2$ hence $q_n^n > e^{-2n(1-q_n)} > e^{-2c}$ and P_2 follows. ■

Theorem 3 *If the function f is continuous at the point $x \in]0,1[$, then,*

$$\lim_{n\to\infty} M_{n,q_n}^{\alpha,\beta} f(x) = f(x) \tag{7}$$

in the two following cases :

1. *If f is bounded on $[0,1]$ and the sequence (q_n) is such that $\lim_{n\to\infty} q_n = 1$,*
2. *If there exist real numbers $\alpha', \beta' \geq 0$ and a real $\kappa' > 0$ such that, for any $x \in]0,1[$, $\left|x^{\alpha'}(1-x)^{\beta'} f(x)\right| \leq \kappa'$, $\alpha' < \alpha + 1$, $\beta' < \beta + 1$ and the sequence (q_n) owns the property S.*

Theorem 4 *If the function f admits a second derivative at the point $x \in]0,1[$ then, in the cases 1 and 2 of theorem 3, we have the Voronovskaya-type limit :*

$$\lim_{n\to\infty} [n]_{q_n} \left(M_{n,q_n}^{\alpha,\beta} f(x) - f(x)\right) = \frac{d}{dx}\left(x^{\alpha+1}(1-x)^{\beta+1} f'(x)\right)/x^{\alpha}(1-x)^{\beta}\,. \tag{8}$$

(The limit operator is the Jacobi differential operator for the weight $x^{\alpha}(1-x)^{\beta}$)

For the proofs of theorems 3 and 4 we need some preparation.

Proposition 2 *We set, for any $n, m \in \mathbb{N} - \{0\}$ and $x \in [0,1]$, $q \in [1/2, 1[$,*

$$T_{n,m,q}(x) = \sum_{k=0}^{n} b_{n,k,q}(x) \frac{\displaystyle\int_0^1 t^{k+\alpha}\ (1-qt)_q^{n-k+\beta}\ (x-t)^m d_q t}{\displaystyle\int_0^1 t^{k+\alpha}\ (1-qt)_q^{n-k+\beta} d_q t}. \tag{9}$$

For any m, there exists a constant $K_m > 0$, independent of n and q, such that :

$$\sup_{x\in[0,1]} |T_{n,m,q}(x)| \leq \begin{cases} K_m/\left[n\right]_q^{m/2} & \text{if } m \text{ is even,} \\ K_m/\left[n\right]_q^{(m+1)/2} & \text{if } m \text{ is odd.} \end{cases}$$

To prove this proposition we consider the lemmas 2 and 3.

Lemma 2 *We set, for any $n, m \in \mathbb{N}$ and $x \in [0,1]$,*

$$T^1_{n,m,q}(x) = \sum_{k=0}^{n} b_{n,k,q}(x) \frac{\int_0^1 t^{k+\alpha}\ (1-qt)_q^{n-k+\beta}\ (x-t)_q^m d_q t}{\int_0^1 t^{k+\alpha}\ (1-qt)_q^{n-k+\beta} d_q t}.$$

The following recursion formula holds for any $q \in [1/2, 1[$ and $m \geq 2$:

$$[n+m+\alpha+\beta+2]_q\, q^{-\alpha-2m-1} T^1_{n,m+1,q}(x) =$$

$$(-x(1-x)D_q T^1_{n,m,q}(x) + T^1_{n,m,q}(x)(p_{1,m}(x) + x(1-q)\,[n+\alpha+\beta]_q\,[m+1]_q\, q^{1-\alpha-m})$$

$$+T^1_{n,m-1,q}(x)p_{2,m}(x) + T^1_{n,m-2,q}(x)p_{3,m}(x)(1-q), \qquad (10)$$

where the polynomials $p_{i,m}(x)$, $i = 1, 2$ and 3 are uniformly bounded with regard to n and q.

Proof. 1) We set $\psi_k(t) = t^{k+\alpha}(1-qt)_q^{n-k+\beta}$ and $l_k(x) = b_{n,k}(x) \Big/ \int_0^1 \psi_k(t) d_q t$, $k = 0, \ldots, n$ and $T^1_{n,m,q} = T^1_m$.

We compute $x(1-x)D_q T^1_m(x) = x(1-x)\,[m] \sum_{k=0}^n l_k(x) \int_0^1 \psi_k(t)(x-t)_q^{m-1} d_q t$ $+ \sum_{k=0}^n l_k(x) \int_0^1 \psi_k(t)(qx-t)_q^m([k]-[n]\,x)d_q t = A + B.$

We have $A = x(1-x)\,[m]\,T^1_{m-1}(x)$ and

$B = q^{-\alpha} \sum_{k=0}^n l_k(x) \int_0^1 (D_q\psi_k)(t)\, t(1-qt)(qx-t)_q^m d_q t$

$-q^{1-\alpha-m}\,[n+\alpha+\beta] \sum_{k=0}^n l_k(x) \int_0^1 \psi_k(t)(qx-t)_q^{m+1} d_q t$

$+(x([n+\alpha+\beta]\,q^{2-\alpha-m} - [n]) + [-\alpha]) \sum_{k=0}^n l_k(x) \int_0^1 \psi_k(t)(qx-t)_q^m d_q t$

$= B_1 + B_2 + B_3.$

We q-integrate by parts, setting $\sigma(t) = \left(\frac{t}{q}(1-t)(qx-\frac{t}{q})_q^m\right)$. The q-integral in B_1 becomes $\int_0^1 D_q\psi_k(t)\, t(1-qt)(qx-t)_q^m d_qt = [\psi_k(t)\sigma(t)]_0^1 - \int_0^1 \psi_k(t)(D_q\sigma)(t)d_qt$ for each $k = 0, \dots, n$,

We expand $\sigma(t) = q^{-2m}(x-\frac{t}{q})_q^{m+2} + q^{-2m}([3]-q^{m+2})x - q^m)(x-\frac{t}{q})_q^{m+1}$

$+q^{-2m+1}(x(q^{m-1}+[m]\,(1-q)q^m) - x^2(1+[2]\,q(1-q)\,[m])(x-\frac{t}{q})_q^m$

$-q^{2m+3}x^2\,[m]\,(1-q)(q^{m-2}-x)(x-\frac{t}{q})_q^{m-1}$.

We obtain $B_1 = -q^{-\alpha-2m-1}([m+2]\,T^1_{m+1}(x) - [m+1]\,([3]\,x - q^{m+2}x - q^m)T^1_m(x)$

$-q^{-\alpha-2m}\,[m]\,x(q^{m-1}+(1-q)\,[m]\,q^m - x(1+q(1-q)\,[2]\,[m])T^1_{m-1}$

$+q^{-\alpha+2-2m}\,[m-1]\,x^2\,[m]\,(q^{m-2}-x)(1-q)T^1_{m-2}(x)$.

Moreover we have $B_2 = -q^{1-\alpha-m}\,[n+\alpha+\beta]\,(T^1_{n,m+1}(x) - (1-q)\,[m+1]\,xT^1_{n,m}(x))$,

$B_3 = (x(q^n\,[\beta-m+2] - [2-\alpha-m]) + [-\alpha])(T^1_{n,m}(x) - (1-q)\,[m]\,xT^1_{n,m-1}(x))$. ■

Lemma 3 *For any $m \in \mathbb{N}$, $q \in [1/2, 1[$, $x \in [0,1]$, the expansion of $(x-t)^m$ on the Newton basis at the points x/q^i, $i = 0, \dots, m-1$ is:*

$$(x-t)^m = \sum_{k=1}^{m} d_{m,k}(1-q)^{m-k}(x-t)_q^k, \tag{11}$$

where the coefficients $d_{m,k}$, verify $|d_{m,k}| \le d_m$, $k = 1, \dots, m$, and d_m does not depend on x, t, q.

Proof. For $m = 1$, it is obvious. If for some $m \ge 1$, the relation (11) holds, we write $x - t = q^{-k}((x-q^kt) - (1-q)\,[k]\,x)$ for $k = 1, \dots, m$ and we obtain $(x-t)^{m+1}$ $= \sum_{k=1}^{m+1} d_{m+1,k}(1-q)^{m+1-k}(x-t)_q^k$ with $d_{m+1,k} = q^{-k}(qd_{m,k-1} - [k]\,d_{m,k})$ if $k = 1, \dots, m$ and $d_{m+1,m+1} = q^{-m}d_{m,m}$. Since $|d_{m,k}| \le d_m$ we have $|d_{m+1,k}| \le d_{m+1} = 2^{-m}(m+1)d_m$, $k = 1, \dots, m+1$. ■

Proof of the proposition 2

At first we prove that, for any x, $|T_m^1(x)| \leq H_m/[n]^{m/2}$ if m is even (respectively $\leq H_m/[n]^{(m+1)/2}$ if m is odd), where H_m does not depend on n, q, x. We have $T_{n,0}^1(x) = 1$ and the formulae for $M_n^{\alpha,\beta} f_i, i = 1, 2$ of the proof of theorem 2 give the result for $m = 1$ and 2. The product $[n + \alpha + \beta](1 - q) = 1 - q^{n+\alpha+\beta}$ is positive and bounded by $\max(1, \left|1 - 2^{-(1+\alpha+\beta)}\right|)$. If the result is true for some $p \geq 2$, p odd (respectively even) and any $m \leq p$, the result for $p + 1$ follows from the recursion formula (10) of lemma 2.

Then, we write, for any $n, m \in \mathbb{N}$ and $x \in [0, 1]$, using lemma 3, and $1-q < 1/[n]$,

$|T_{n,m,q}(x)| \leq d_m \sum_{k=1}^{m}(1 - q)^{m-k} |T_k^1(x)| \leq d_m(T_m^1(x) + \sum_{k=1}^{m-1} [n]^{-m+k} H_k [n]^{-k/2}$

$\leq d_m T_m^1(x) + \sum_{k=1}^{m-1} H_k [n]^{-(m+1)/2})$ and the result follows.■

Now the following lemma is the key.

Lemma 4 *Let* (q_n) *be a sequence owning the property* S*,* $x \in]0, 1[$ *and* $\delta \in]0, 1[$*,* $\delta < \min(x, 1 - x)$*. Let* $\alpha, \beta, \alpha', \beta'$ *be real numbers such that* $\alpha', \beta' \geq 0$*,* $\alpha > \alpha' - 1$*,* $\beta > \beta' - 1$*. We set* $\varphi(t) = t^{-\alpha'}(1 - t)^{-\beta'}$*,* $t \in]0, 1[$ *and* $I_{x,\delta}(t) = 1$ *if* $|t - x| > \delta$*,* $I_{x,\delta}(t) = 0$ *elsewhere.*

The sequence $E_n(x, \delta) = \sum_{k=0}^{n} b_{n,k,q_n}(x) \dfrac{\int_0^1 t^{k+\alpha}(1 - q_n t)_{q_n}^{n-k+\beta} \varphi(q_n^{\beta+1} t) I_{x,\delta}(t) d_{q_n} t}{\int_0^1 t^{k+\alpha}(1 - q_n t)_{q_n}^{n-k+\beta} d_{q_n} t}$.

verifies $\lim_{n\to\infty} n E_n(x, \delta) = 0$ *for any* x *and* δ *such that* $0 < \delta < x < 1 - \delta$*.*

Proof. Let $\overline{\alpha}$ (respectively $\overline{\beta}$) be the smallest integer such that $\overline{\alpha} \geq \alpha$ (respectively $\overline{\beta} \geq \beta$) and τ (respectively τ') be a real number such that $\tau > \dfrac{\overline{\alpha} + \overline{\beta} + 2}{\alpha - \alpha' + 1}$ (respectively $\tau' > \dfrac{\overline{\alpha} + \overline{\beta} + 2}{\beta - \beta' + 1}$).

For any $k=0,\dots,n$, we have $\int_0^1 t^{k+\alpha}(1-qt)_q^{n-k+\beta}d_qt \geq \int_0^1 t^{k+\overline{\alpha}}(1-qt)_q^{n-k+\overline{\beta}}d_qt$ $= \frac{[k+\overline{\alpha}]![n-k+\overline{\beta}]!}{[n+\overline{\alpha}+\overline{\beta}+1]!} \geq \begin{bmatrix} n \\ k \end{bmatrix}^{-1}(n+\overline{\alpha}+\overline{\beta}+1)^{-(\overline{\alpha}+\overline{\beta}+1)})$. We set $I^-_{x,\delta}(t)=1$ if $0<t<x-\delta$ and $I^+_{x,\delta}(t)=1$ if $x+\delta<t<1$, $I^-_{x,\delta}(t)=I^+_{x,\delta}(t)=0$ elsewhere.

We split the interval $(0,1)$ introducing $e_n=1/n^\tau$, $e'_n=1-1/n^{\tau'}$, $n\in\mathbb{N}$, and we define, using again $\psi^{\alpha,\beta}_{n,k,q}(t)=t^{k+\alpha}(1-qt)_q^{n-k+\beta}$ and $l_{n,k,q}(x)=b_{n,k,q}(x)\Big/\int_0^1 \psi^{\alpha,\beta}_{n,k,q}(t)d_qt$ of lemma 2, $A^1_n=\sum_{k=0}^n l_{n,k,q_n}(x)\int_0^{e_n} t^{k+\alpha-\alpha'}(1-q_nt)_{q_n}^{n-k+\beta}d_{q_n}t$,

$A^2_n=\sum_{k=0}^n l_{n,k,q_n}(x)\int_{e_n}^1 t^{k+\alpha-\alpha'}(1-q_nt)_{q_n}^{n-k+\beta}I^-_{x,\delta}(t)d_{q_n}t$,

$A^3_n=\sum_{k=0}^n l_{n,k,q_n}(x)\int_0^{e'_n} t^{k+\alpha}(1-q_nt)_{q_n}^{n-k+\beta}\big/(1-q_n^{\beta+1}t)^{\beta'}\ I^+_{x,\delta}(t)d_{q_n}t$,

$A^4_n=\sum_{k=0}^n l_{n,k,q_n}(x)\int_{e'_n}^1 t^{k+\alpha}(1-q_nt)_{q_n}^{n-k+\beta}\big/(1-q_n^{\beta+1}t)^{\beta'}\ I^+_{x,\delta}(t)d_qt$.

If $t>x$, (respectively if $t<x$) then $t^{-\alpha'}<x^{-\alpha'}$ (respectively $(1-q_n^{\beta+1}t)^{\beta'}$ $>(1-q_n^{\beta+1}x)^{\beta'}\geq(1-x)^{\beta'}$).

Hence, we have $E_n(x,\delta)\leq(1/2)^{-(\beta+1)\alpha'}((1-x)^{-\beta'}(A^1_n+A^2_n)+x^{-\alpha'}(A^3_n+A^4_n))$ if $q_n\geq1/2$, and it is sufficient to prove $\lim\limits_{n\to\infty} nA^i_n=0$ for $i=1,2,3,4$.

If $q_n^n\geq c$ and $e_n<1/2$, we have for $k=0,\dots,n$, $\int_0^{e_n} t^{k+\alpha-\alpha'}(1-q_nt)_{q_n}^{n-k+\beta}d_{q_n}t$ $=q_n^{-k(\beta+1)}\begin{bmatrix} n \\ k \end{bmatrix}_{q_n}^{-1}\int_0^{e_n} b_{n,k,q}(q_n^{\beta+1}t)t^{\alpha-\alpha'}(1-q_nt)_{q_n}^{\beta}d_{q_n}t\leq\begin{bmatrix} n \\ k \end{bmatrix}_{q_n}^{-1}\gamma_1e_n^{\alpha-\alpha'+1}$, where γ_1 does not depend on k,n,x, since $q_n^{(\beta+1)k}\geq c^{\beta+1}$, $0\leq b_{n,k,q_n}(q_n^{\beta+1}t)\leq1$, and, $(1-q_nt)_{q_n}^\beta\leq1$ if $\beta\geq0$ and $t\in[0,1]$, (respectively $(1-q_nt)_{q_n}^\beta\leq(1-e_n)^{-1}\leq2$ if $\beta<0$ and $t\in[0,e_n]$). Hence, we have $A^1_n\leq\gamma_1(n+\overline{\alpha}+\overline{\beta}+1)^{\overline{\alpha}+\overline{\beta}+1}n^{-\tau(\alpha-\alpha'+1)}$. The choice of τ and the property S on (q_n) give $\lim\limits_{n\to\infty} nA^1_n=0$.

We choose $m\in\mathbb{N}$ such that $m>\tau\alpha'+1$ and we write

$A^2_n\leq n^{\tau\alpha'}\delta^{-2m}\sum\limits_{k=0}^n l_{n,k,q_n}(x)\int_{e_n}^1 t^{k+\alpha}(1-q_nt)_{q_n}^{n-k+\beta}(x-t)^{2m}d_{q_n}t$

$\leq n^{\tau\alpha'}\delta^{-2m}T_{n,2m,q_n}(x)\leq K_{2m}\delta^{-2m}n^{\tau\alpha'-m}$, hence $\lim\limits_{n\to\infty} nA^2_n=0$ by the choice of m.

Now we have, if $t<e'_n$, $(1-q_n^{\beta+1}t)^{\beta'}>(1-e'_n)^{\beta'}\geq n^{-\tau'\beta'}$, hence

$A_n^3 \leq n^{\tau'\beta'} \sum\limits_{k=0}^{n} l_{n,k,q_n}(x) \int_0^{e'_n} t^{k+\alpha}(1-q_n t)_{q_n}^{n-k+\beta} I_{x,\delta}^{+}(t) d_{q_n} t$. We choose $m' \in \mathbb{N}$ such that $m' > \tau'\beta' + 1$ to have $A_n^3 \leq n^{\tau'\beta'} \delta^{-2m'} \sum\limits_{k=0}^{n} l_{n,k,q_n}(x) \int_0^1 t^{k+\alpha}(1-q_n t)_{q_n}^{n-k+\beta}(x-t)^{2m'} d_{q_n} t$ $\leq n^{\tau'\beta'} \delta^{-2m'} T_{n,2m',q_n} \leq K_{2m'} \delta^{-2m'} n^{\tau'\beta'-m'}$, hence $\lim\limits_{n\to\infty} nA_n^3 = 0$ by the choice of m'.

To finish, we prove that $(1-q_n^{\beta-\beta'+1}t)_{q_n}^{\beta'} \leq (1-q_n^{\beta+1}t)^{\beta'}$ for any $t \in [0,1]$.

If $0 \leq \beta' < 1$, we use the q-binomial formula (cf. [1]) and the inequalities $[-\beta'] \leq -\beta'$ and $[-\beta'+k]/[k] \geq (-\beta'+k)/k$ for any integer $k \geq 1$. In the other cases, if l is the integer such that $l \leq \beta' < l+1$, we use the rules of product of q-binomials to write

$(1-q_n^{\beta-\beta'+1}t)_{q_n}^{\beta'} = (1-q_n^{\beta-\beta'+1}t)_{q_n}^{l}(1-q_n^{\beta-\beta'+1+l}t)_{q_n}^{\beta'-l}$ and the result follows.

Then, with the same rules, we write, for any $k = 0, \ldots, n$ and $t \in [0,1]$,

$(1-q_n t)_{q_n}^{n-k+\beta} = (1-q_n t)_{q_n}^{\beta-\beta'}(1-q_n^{\beta-\beta'+1}t)_{q_n}^{\beta'}(1-q_n^{\beta+1}t)_{q_n}^{n-k}$ and

$(1-q_n t)_{q_n}^{n-k+\beta} / (1-q_n^{\beta+1}t)^{\beta'} \leq (1-q_n t)_{q_n}^{\beta-\beta'}(1-q_n^{\beta+1}t)_{q_n}^{n-k}$. We deduce if $q_n^n \geq c$ and $e'_n > 1/2$, $A_n^4 \leq \sum\limits_{k=0}^{n} q_n^{-(\beta+1)k} \begin{bmatrix} n \\ k \end{bmatrix}_{q_n}^{-1} l_{n,k,q_n}(x) \int_{e'_n}^{1} t^{\alpha}(1-q_n t)_{q_n}^{\beta-\beta'} b_{n,k,q_n}(q_n^{\beta+1}t) d_{q_n} t$ $\leq \gamma_2(n+\overline{\alpha}+\overline{\beta}+1)^{\overline{\alpha}+\overline{\beta}+1}(1-e'_n)^{\beta-\beta'+1}$ where γ_2 does not depend on k, n, x. The choice of e'_n and τ' gives $\lim\limits_{n\to\infty} nA_n^4 = 0$. ∎

<u>Proof of theorem 3</u>

Suppose f is continuous at $x \in]0,1[$. Let $\varepsilon > 0$ be an arbitrary real number. There exists $\delta' > 0$ such that $|f(x) - f(t)| < \varepsilon$ for any $t \in [0,1]$ such that $|x-t| < \delta'$. Let $\delta = \delta'/2$ and $N' \in \mathbb{N}$ such that $(1-q_n^{\beta+1})x < \delta$ for $n > N'$. Then we have, if $|x-t| < \delta$ and $n > N'$, the inequalities $-\delta < -q_n^{\beta+1}\delta < x - q_n^{\beta+1}t = q_n^{\beta+1}(x-t) + (1-q_n^{\beta+1})x < 2\delta$ and $\left|f(x) - f(q_n^{\beta+1}t)\right| < \varepsilon$.

Hence, we have, if $|f|$ is bounded by κ, $\left|f(x) - f(q_n^{\beta+1}t)\right| < \varepsilon + 2\kappa I_{x,\delta}(t)$ and, in

the case 2, $\left|f(x)-f(q_n^{\beta+1}t)\right| < \varepsilon + (|f(x)| + \kappa'(q_n^{\beta+1}t)^{-\alpha'}(1-q_n^{\beta+1}t)^{-\beta'} I_{x,\delta}(t)$.

We apply the operator $M_{n,q_n}^{\alpha,\beta}$ at the function $h_x(t) = f(t) - f(x)$.

We have $\left|M_{n,q_n}^{\alpha,\beta} f(x) - f(x)\right| = \left|M_{n,q_n}^{\alpha,\beta} h_x(x)\right| \leq \left(M_{n,q_n}^{\alpha,\beta} |h_x|\right)(x)$

$$\leq \begin{cases} \varepsilon + 2\kappa T_{n,2,q_n}(x)/\delta^2 \text{ in the case 1,} \\ \varepsilon + |f(x)|\, T_{n,2,q_n}(x)/\delta^2 + \kappa' E_n(x,\delta,q_n) \text{ in the case 2.} \end{cases}$$

The second term (respectively and the third term in the case 2) of the right hand side vanishes when $[n]_{q_n}$ tends to infinity. Since $\lim\limits_{n\to\infty} 1/[n]_{q_n} = 0$ in both cases (remark 1), the result follows.■

<u>Proof of theorem 4</u>

We write Taylor formula at the point x,

$f(t) = f(x) + (t-x)f'(x) + (t-x)^2 f''(x)/2 + (t-x)^2\varepsilon(t-x)$ where $\lim\limits_{u\to 0}\varepsilon(u) = 0$.

We apply the operator $M_{n,q_n}^{\alpha,\beta}$ at the function f of the variable t to obtain

$M_{n,q_n}^{\alpha,\beta} f(x) - f(x) = -f'(x)T_{n,1,q_n}(x) + \dfrac{f''(x)}{2}T_{n,2,q_n}(x) + R_n(x)$ where

$R_n(x) = M_{n,q_n}^{\alpha,\beta}\zeta_x(x)$ with $\zeta_x(t) = (t-x)^2\varepsilon(t-x)$. We use $\lim\limits_{q\to 1}[a]_q = a$ for any $a \in \mathbb{R}$ and we verify, with the help of the formulae of the proof of theorem 2, that

$\lim\limits_{[n]_{q_n}\to\infty} [n]_{q_n} T_{n,1,q_n}(x) = (\alpha+\beta+2)x - \alpha - 1$ and $\lim\limits_{[n]_{q_n}\to\infty} [n]_{q_n} T_{n,2,q_n}(x) = 2x(1-x)$.

So, to obtain the result, we have to prove that $\lim\limits_{[n]_{q_n}\to\infty} [n]_{q_n} R_n(x) = 0$. We proceed in the same manner as in the proof of theorem 3. For any arbitrary $\eta > 0$, we can find $\delta > 0$ such that, for n great enough, $\varepsilon(x-t) < \eta$ if $\left|x - q_n^{\beta+1}t\right| < \delta$.

We obtain the inequality $|\zeta_x(t)| \leq \eta(x-t)^2 + (\rho_x + |f(t)|)I_{x,\delta}(q^{-(\beta+1)}t)$ for any $t \in]0,1[$, where ρ_x is independent of t and δ. We deduce

$$[n]_{q_n}|R_n(x)| \leq \begin{cases} [n]_{q_n}(\eta T_{n,2,q_n}(x) + (\rho_x+\kappa)T_{n,4,q_n}(x)/\delta^4) \text{ in the case 1,} \\ [n]_{q_n}(\eta T_{n,2,q_n}(x) + \rho_x T_{n,4,q_n}(x)/\delta^4) + \kappa' n E_n(x,\delta))\text{in the case 2.} \end{cases}$$

The right hand side tends to $2\eta x(1-x)$ when n (hence $[n]_{q_n}$) tends to infinity and is as small as wanted.$\blacksquare$

Remark 2

1) We see that the best order of approximation in (8) is in $1/\left[n\right]_{q_n}$. If $1-q_n=1/n^{\gamma}$ with $0<\gamma<1$, then $\lim\limits_{n\to\infty}\left[n\right]_{q_n}/n^{\gamma}=1$, hence $\left[n\right]_{q_n}$ can be replaced by n^{γ} in (8). If $1-q_n=1/n\log n$ or $1/n^{\gamma}$ with $\gamma>1$, then $\lim\limits_{n\to\infty}\left[n\right]_{q_n}/n=1$, $\left[n\right]_{q_n}$ can be replaced by n and we refound exactly the Voronovskaya-limit property of $M_{n,1}^{\alpha,\beta}(x)$ (case 1).

2) In the case 2, the theorems 3 and 4 are valid for $M_{n,1}^{\alpha,\beta}$, if wf is Lebesgue integrable on $[0,1]$, and this result is new. (In the proof the Jackson integrals have to be replaced by Lebesgue integrals)

Theorem 5 *If f' is continuous on $[0,1]$ and $q>1/2$, then*

$$\left\|D_q(M_{n,q}^{\alpha,\beta}f)-f'\right\|_{\infty}\leq C'_{\alpha,\beta}\left(\omega\left(f',\frac{1}{\sqrt{\left[n\right]_q}}\right)+\omega\left(f',1-q\right)\right)+\frac{\left[\alpha+\beta+2\right]_q}{\left[n\right]_q}\left\|f'\right\|_{\infty},$$

where $C'_{\alpha,\beta}$ is a constant independent of n,q,f.

Hence, if $\lim\limits_{n\to\infty}q_n=1$, then $\lim\limits_{n\to\infty}D_{q_n}(M_{n,q_n}^{\alpha,\beta}f)=f'(x)$ uniformly on $[0,1]$.

Proof. We write, using (5), for any $x\in[0,1]$,

$DM_n^{\alpha,\beta}f(x)-f'(x)=\frac{[n]}{[n+\alpha+\beta+2]}\left(M_{n-1}^{\alpha+1,\beta+1}\left(Df\left(\frac{\cdot}{q}\right)\right)(qx)-Df(x)+Df(x)-f'(x)\right)$ $+\left(\frac{[n]}{[n+\alpha+\beta+2]}-1\right)f'(x)$. Since $0<\frac{[n]}{[n+\alpha+\beta+2]}<1$, we have $\left|D(M_n^{\alpha,\beta}f(x))-f'(x)\right|$ $\leq\left|M_{n-1}^{\alpha+1,\beta+1}\left(Df\left(\frac{\cdot}{q}\right)\right)(qx)-Df(x)\right|+\left|Df(x)-f'(x)\right|+\frac{[\alpha+\beta+2]}{[n]}\left|f'(x)\right|.$

The theorem (2) for the function $Df\left(\frac{\cdot}{q}\right)$ gives

$\left|M_{n-1}^{\alpha+1,\beta+1}\left(Df\left(\frac{\cdot}{q}\right)\right)(qx)-Df(\frac{\cdot}{q})(qx)\right|\leq C_{\alpha+1,\beta+1}\omega\left(Df(\frac{\cdot}{q}),\frac{1}{\sqrt{[n-1]}}\right)$. Moreover

$|Df(x)-f'(x)|=|f'(y)-f'(x)|$ for some y with $qx<y<x$ hence $|y-x|<1-q$ and $|Df(x)-f'(x)|\le \omega(f',1-q)$. The modulus of continuity of $Df\left(\frac{\cdot}{q}\right)$ is linked with the modulus of continuity of f'. Indeed, for any $y_i\in[0,1]$ and $i=1,2$, there exists z_i, such that , $y_i<z_i<y_i/q$ and $Df(\frac{\cdot}{q})(y_i)=f'(z_i)$. As $|z_1-z_2|\le |y_1-y_2|/q+(1-q)/q$ we get $\omega\left(Df(\frac{\cdot}{q}),t\right)=\sup\limits_{|y_1-y_2|<t}\left|Df(\frac{\cdot}{q})(y_1)-Df(\frac{\cdot}{q})(y_2)\right|\le \sup\limits_{|y_1-y_2|<t}(|f'(z_1)-f'(z_2)|$ $\le 2(\omega(f',t)+\omega(f',1-q))$ and the result follows. ∎

Corollary 3 *If f' is continuous on $[0,1]$ and $1-q_n=o(1/n^4)$, then*

$$\lim_{n\to\infty}\left(M^{\alpha,\beta}_{n,q_n}f\right)'(x)=f'(x) \text{ uniformly on } [0,1].$$

Proof. For any $x\in[0,1]$, there exists $u\in(q_nx,x)$ such that

$D_{q_n}(M^{\alpha,\beta}_{n,q_n}f)(x)=(M^{\alpha,\beta}_{n,q_n}f)'(u)$ and $(x-u)<1-q_n$.

Hence $\left|D_{q_n}(M^{\alpha,\beta}_{n,q_n}f)(x)-(M^{\alpha,\beta}_{n,q_n}f)'(x)\right|\le(1-q_n)\left|(M^{\alpha,\beta}_{n,q_n}f)''(v)\right|$ for some v and $\left\|D_{q_n}(M^{\alpha,\beta}_{n,q_n}f)-(M^{\alpha,\beta}_{n,q_n}f)'\right\|_\infty\le n^4(1-q_n)\left\|M^{\alpha,\beta}_{n,q_n}f\right\|_\infty\le n^4(1-q_n)\|f\|_\infty$, via Markov inequality. ∎

4 self-adjointness properties

In this part $q\in]0,1[$ is independent of n.

On the space of polynomials $\langle.,.\rangle^{\alpha,\beta}_q$ is an inner product. Let $\left(P^{\alpha,\beta}_{r,q}\right)_{r\in\mathbb{N}}$ be the sequence of the orthogonal polynomials for $\langle.,.\rangle^{\alpha,\beta}_q$ such that degree of $P^{\alpha,\beta}_{r,q}=r$ and $P^{\alpha,\beta}_{r,q}(0)=\begin{bmatrix} r+\alpha \\ r\end{bmatrix}_q=\dfrac{[\alpha+r]_q\ldots[\alpha+1]_q}{[r]_q!}$. We set $\nu_r=\left(\langle P^{\alpha,\beta}_{r,q},P^{\alpha,\beta}_{r,q}\rangle^{\alpha,\beta}_q\right)^{1/2}$.

We define $U^{\alpha,\beta}_q$ which is a q-analogue of the Jacobi differential operator by:

$$U^{\alpha,\beta}_q f(x)=D_q\left(x^{\alpha+1}(1-q^{-\beta-1}x)^{\beta+1}_q D_q f(\frac{x}{q})\right)/x^\alpha(1-q^{-\beta}x)^\beta_q\,. \tag{12}$$

Proposition 3 *The operator $U_q^{\alpha,\beta}$ is self-adjoint for $\langle .,.\rangle_q^{\alpha,\beta}$. It preserves the space of polynomials of degree $r \in \mathbb{N}$. Consequently, for any $r \in \mathbb{N}$, $P_{r,q}^{\alpha,\beta}$ is eigenvector of $U_q^{\alpha,\beta}$ for the eigenvalue $\mu_r^{\alpha,\beta} = -q^{-\beta-r}\,[r]_q\,[r+\alpha+\beta+1]_q$.*

Proof.

$U^{\alpha,\beta}$ is a q-differential operator of order 2. We compute with q-binomial relations, $U^{\alpha,\beta}f(x) = \big(-q^{\alpha-\beta}\,[\beta+1]\,x + [\alpha+1]\,(1-q^{-\beta-1}x)\big)Df(x) + (1-q^{-\beta-1}x)\frac{x}{q}D^2f(\frac{x}{q})$, hence the operator U preserves the degree of polynomials. If f and g are polynomials $\langle Uf, g\rangle$ is well defined. We write, since the q-integration by parts is valid, $\langle U^{\alpha,\beta}f, g\rangle$ $= \Big[x^{\alpha+1}(1-q^{-\beta-1}x)_q^{\beta+1}Df(\frac{x}{q})g(x)\Big]_0^{q^{\beta+1}} - \int_0^{q^{\beta+1}}(qx)^{\alpha+1}(1-q^{-\beta}x)_q^{\beta+1}Df(x)Dg(x)d_qx.$ and the first term vanishes. We compute $U^{\alpha,\beta}f(x)$ for $f(x) = x^r$ to obtain $U^{\alpha,\beta}f(x) = q^{1-r}\,[r]\,([\alpha+r]\,x^{r-1}(1-x) - q^{-\beta-1}\,[\beta+1]\,x^r$ where the coefficient of x^r is the eigenvalue $\mu_r^{\alpha,\beta}$. ■

Proposition 4 *The eigenvectors of the operators $M_{n,q}^{\alpha,\beta}, n \in \mathbb{N}$, are the polynomials $P_{r,q}^{\alpha,\beta}, r \in \mathbb{N}$ and, if f satisfies $c(\alpha)$, we have*
$M_{n,q}^{\alpha,\beta}f = \sum_{r=0}^{n}\lambda_{n,r}^{\alpha,\beta}\langle f, P_{r,q}^{\alpha,\beta}\rangle_q^{\alpha,\beta}P_{r,q}^{\alpha,\beta}/\nu_r^2$ *with the eigenvalues*
$\lambda_{n,r}^{\alpha,\beta} = q^{r(r+\alpha+\beta+1)}\dfrac{[n]!}{[n-r]!}\dfrac{\Gamma_q(n+\alpha+\beta+2)}{\Gamma_q(n+r+\alpha+\beta+2)}$ *if $r \le n$, $\lambda_{n,r}^{\alpha,\beta} = 0$ otherwise.*

Proof. Since M_n is self adjoint and preserve the degree of polynomials, the orthogonal polynomials P_r are eigenvectors. The eigenvalue $\lambda_{n,r}^{\alpha,\beta}$ is obtained by computing $M_nf(x)$ for $f(x) = x^r$.

We use (5) r times to get $D^rM_nf(x) = \dfrac{q^{r(\alpha+\beta+r+1)}\,[r]!\,[n]\ldots[n-r+1]}{[n+\alpha+\beta+2]\ldots[n+r+\alpha+\beta+1]}$. ■

Corollary 4 *1. For any $n, m \in \mathbb{N}$, the operators $M_{n,q}^{\alpha,\beta}$ and $M_{m,q}^{\alpha,\beta}$ commute on the space of functions satisfying $C(\alpha)$.*

2. *For any $n \in \mathbb{N}$, the operators $M^{\alpha,\beta}_{n,q}$ and $U^{\alpha,\beta}_q$ commute on the space of functions f such that f' is defined in a neighborhood of 0 and is continuous at the point 0.*

Proof. 2) For any $r \in \mathbb{N}$ the q-integrals $\langle f, UP_r\rangle$ and $\langle Uf, P_r\rangle$ are well defined if f' is continuous at the point 0. We go from one to the other by two q-integrations by parts which are valid because $\lim\limits_{x\to 0} Df(\frac{x}{q}) = f'(0)$. Then we write $UM_n f = \sum\limits_{r=0}^{n} \lambda_{n,r}\langle f, P_r\rangle \mu_r P_r/\nu_r^2 = \sum\limits_{r=0}^{n} \lambda_{n,r}\langle Uf, P_r\rangle P_r/\nu_r^2 = M_n Uf.$ ■

Remark 3

This proposition and its corollary open a field to study $\lim_{n\to\infty} M^{\alpha,\beta}_{n,q} f$ for q fixed. Formally we have $\lim_{n\to\infty} M^{\alpha,\beta}_{n,q} f = S^{\alpha,\beta}_q f = \sum_{r=0}^{\infty} q^{r(r+\alpha+\beta+1)}\langle f, P^{\alpha,\beta}_{r,q}\rangle^{\alpha,\beta}_q P^{\alpha,\beta}_{r,q}/\nu_r^2$ and $\lim_{n\to\infty} M^{\alpha,\beta}_{n,q} Q = \sum_{r=0}^{\deg Q} q^{r(r+\alpha+\beta+1)}\langle Q, P^{\alpha,\beta}_{r,q}\rangle^{\alpha,\beta}_q P^{\alpha,\beta}_{r,q}(x)/\nu_r^2$ if Q is a polynomial. So $\lim_{n\to\infty} M^{\alpha,\beta}_{n,q} f = f$, if and only if f is a constant. Moreover, $\lambda^{\alpha,\beta}_{n-1,r} - \lambda^{\alpha,\beta}_{n,r} = \dfrac{q^{n+\beta}\lambda^{\alpha,\beta}_{n,r}\mu^{\alpha,\beta}_r}{[n]\,[n+\alpha+\beta+1]}$, $r \in \mathbb{N}$, hence $M^{\alpha,\beta}_{n-1} f - M^{\alpha,\beta}_n f = \dfrac{q^{n+\beta}}{[n]\,[n+\alpha+\beta+1]} U^{\alpha,\beta} M^{\alpha,\beta}_n f$ and it is easy to prove (cf. [2]) that, when f' is defined in a neighborhood of 0, continuous at 0, $\left\| M^{\alpha,\beta}_{n,q} f - S^{\alpha,\beta}_q f\right\|_\infty \le \gamma_n \sup_{x\in\left[0,q^{\beta+2}\right]} \left|U^{\alpha,\beta}_q f(x)\right|$, where $\gamma_n = \sum_{k=n+1}^{\infty} \dfrac{q^{k+\beta}}{[k]_q\,[k+\alpha+\beta+1]_q} \sim \dfrac{q^{n+\beta+1}}{[n]_q}$. Of course $U^{\alpha,\beta}_q f$ has to be bounded on $\left[0, q^{\beta+2}\right]$, which is true, for example, if f is bounded on $[0,1]$ and continuous on $[0, A]$ for some $A < 1$.

Proposition 5 *The polynomials $P^{\alpha,\beta}_{r,q}$ are q-extensions of Jacobi polynomials for the weight $x^\alpha(1-x)^\beta$ denoted $P^{\alpha,\beta}_r, r \in \mathbb{N}$. They own the following properties which are the q-analogues of the well-known properties of Jacobi polynomials.*

1. *For any $r \in \mathbb{N}$, $\lim\limits_{q\to 1} P^{\alpha,\beta}_{r,q} = P^{\alpha,\beta}_r$,*

2. *For any $r \in \mathbb{N}$, the polynomial $P_{r,q}^{\alpha,\beta}$ is a q-hypergeometric function (cf. [7]) :*

$$P_{r,q}^{\alpha,\beta}(x) = \begin{bmatrix} \alpha+r \\ r \end{bmatrix} {}_2\Phi_1 \left[\begin{array}{ccc} q^{-r},\, q^{r+\alpha+\beta+1} & & \\ & ;\ q\,, & q^{-\beta}x \\ q^{\alpha+1} & & \end{array} \right]$$

So we have $P_{r,q}^{\alpha,\beta}(x) = \begin{bmatrix} \alpha+r \\ r \end{bmatrix} p_r(q^{-\beta-1}x; q^{\alpha+1}, q^{\beta+1} : q)$, where $p_r(x; u, v : q)$ is the shifted little q-Jacobi polynomial of degree r (cf. [1], p.592).

3. *They verify a q-analogue of Rodrigues formula :*

$$P_{r,q}^{\alpha,\beta}(x) = \frac{1}{[r]!} \frac{D_q^r \left(x^{\alpha+r}(1 - q^{-\beta-r}x)_q^{\beta+r} \right)}{x^\alpha (1 - q^{-\beta}x)_q^\beta}.$$

4. *We have the relation for the q-derivative :*

$$D_q P_{r,q}^{\alpha,\beta}\left(\frac{\cdot}{q}\right) = -q^{-\beta-r}\,[r + \alpha + \beta + 1]\, P_{r-1,q}^{\alpha+1,\beta+1}.$$

Proof. 2) We look for the analytic solutions of the equation $U_q^{\alpha,\beta} f - \mu_{r,q}^{\alpha,\beta} f = 0$. We write $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $U_q^{\alpha,\beta} f(x) - \mu_{r,q}^{\alpha,\beta} f(x) = [\alpha + 1]\, a_1 - \mu_{r,q}^{\alpha,\beta} a_0$ $+q^{-\beta} \sum_{k=1}^{\infty} \left([k+1]\,[k+\alpha+1]\, q^\beta a_{k+1} - ([k]\,[k+\alpha+\beta+1] - \mu_{r,q}^{\alpha,\beta} q^{k+\beta}) a_k \right) q^{-k} x^k$. We obtain $\dfrac{a_{k+1}}{a_k} = -q^{-r-\beta} \dfrac{([r]-[k])([k+r+\alpha+\beta+1]}{[k+1]\,[k+\alpha+1]} = q^{-\beta} R(q^k)$, for any $k \in \mathbb{N}$, with $R(t) = \dfrac{(q^{-r} - t^{-1})(q^{r+\alpha+\beta+1} - t^{-1})}{(q - t^{-1})(q^{\alpha+1} - t^{-1})}$ and the result follows.

3) For any polynomial Q of degree $< r$, we verify, with the help of q-integrations by parts that $\langle \frac{D_q^r \left(x^{\alpha+1}(1-q^{-\beta-r}x)_q^{\beta+r} \right)}{x^\alpha (1-q^{-\beta}x)_q^\beta}, Q \rangle_q^{\alpha,\beta} = 0$.

We compute $P_{r,q}^{\alpha,\beta}(0)$ by using a q-extension of Leibniz formula. We write $D_q^r \left(x^{\alpha+1}(1 - q^{-\beta-r}x)_q^{\beta+r} \right) = \sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix} \frac{\Gamma_q(\alpha+r+1)\Gamma_q(\beta+r+1)}{\Gamma_q(\alpha+k+1)\Gamma_q(\beta+r-k+1)} x^{\alpha+k} (1 - q^{-\beta}x)^{\beta+r-k} q^{c_k}$, where $c_k = k(k+\alpha-\beta-r+(k-1)/2)$ and $D_q^r \left(x^{\alpha+1}(1 - q^{-\beta-r}x)_q^{\beta+r} \right) / x^\alpha (1 - q^{-\beta}x)_q^\beta =$ $\sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix} \frac{\Gamma_q(\alpha+r+1)\Gamma_q(\beta+r+1)}{\Gamma_q(\alpha+k+1)\Gamma_q(\beta+r-k+1)} x^k (1 - q^{-\beta-r+k}x)^{r-k} q^{c_k} = A(x)$.

We obtain $A(0) = \frac{\Gamma_q(\alpha+r+1)}{\Gamma_q(\alpha+1)} = [r]! \begin{bmatrix} r+\alpha \\ r \end{bmatrix}$ hence $P_{r,q}^{\alpha,\beta}(x) = A(x)/\,[r]!$.

1) We take $\lim\limits_{q\to 1} A(x) = \sum_{k=0}^{r}\binom{r}{k}\frac{\Gamma(\alpha+r+1)\Gamma(\beta+r+1)}{\Gamma(\alpha+k+1)\Gamma(\beta+r-k+1)}x^k(1-x)^{r-k}$. It is $r!P_r^{\alpha,\beta}(x)$ (Rodrigues formula).

4) We use (5) to prove that $D_qP_{r,q}^{\alpha,\beta}(\frac{\cdot}{q})$ is eigenvector of $M_{r-1,q}^{\alpha+1,\beta+1}$. Hence, it is equal to $P_{r-1,q}^{\alpha+1,\beta+1}$ up to a constant. We compute $D_qP_{r,q}^{\alpha,\beta}(0) = a_1 = \mu_{r,q}^{\alpha,\beta}a_0$, hence $a_1 = -\begin{bmatrix} r+\alpha \\ r \end{bmatrix} q^{-\beta-r}[r]\,[r+\alpha+\beta+1]\,/\,[\alpha+1] = -q^{-\beta-r}[r+\alpha+\beta+1]\begin{bmatrix} r+\alpha \\ r-1 \end{bmatrix}$ and the result follows. ∎

5 The case $\alpha = \beta = -1$

In this part, we study the operators $M_{n,q}^{\text{-}1,\text{-}1}$. They are built with $M_{n+1,q}^{0,0}$ as kantorovich operators are built with Bernstein operators (formula (13)).

Definition 3

The operator $M_{n,q}^{\text{-}1,\text{-}1}$ is defined by replacing $\alpha = \beta$ by -1 in formula (1). It is :

$$M_{n,q}^{\text{-}1,\text{-}1}f(x) = \sum_{k=0}^{n} f_{n,k,q}^{\text{-}1,\text{-}1}b_{n,k,q}(x)$$

with $f_{n,0,q}^{\text{-}1,\text{-}1} = f(0)$, $f_{n,n,q}^{\text{-}1,\text{-}1} = f(1)$ and the coefficients $f_{n,k,q}^{\text{-}1,\text{-}1}$, for $k = 1,\dots,n-1$, are given by (3) taking $\alpha = \beta = -1$.
The bilinear form is $\langle f,g\rangle_q^{\text{-}1,\text{-}1} = \displaystyle\int_0^1 \frac{f(t)g(t)}{t(1-t)}d_qt$.

The polynomial $M_{n,q}^{\text{-}1,\text{-}1}f$ is well defined for any function f defined on $[0,1]$, bounded in a neighborhood of 0 (condition C(-1)). It verifies $M_{n,q}^{\text{-}1,\text{-}1}f(0) = f(0)$ and $M_{n,q}^{\text{-}1,\text{-}1}f(0) = f(1)$, hence it preserves the affine functions.

Proposition 6 *If the function f is continuous on* $[0,1]$, *then*

$\lim\limits_{\alpha\to\text{-}1} M_{n,q}^{\alpha,\alpha}f(x) = M_{n,q}^{\text{-}1,\text{-}1}f(x)$ *for any* $x \in [0,1]$.

Proof. The q-binomial coefficients $b_{n,k,q}(x)$ are positive and form a partition of the unity. Hence it is sufficient to prove that $\lim_{\alpha\to-1} f_{n,k,q}^{\alpha,\alpha} = f_{n,k,q}^{\text{-1,-1}}$ for any k. For $k = 1,\dots,n-1$, we compute $f_{n,k,q}^{\alpha,\alpha} - f_{n,k,q}^{\text{-1,-1}} = \frac{\int_0^1 t^{k+\alpha}\ (1-qt)_q^{n-k+\alpha}\ \left(f(q^{\alpha+1}t)-f(t)\right)d_qt}{B_q(k+\alpha+1,n-k+\alpha+1)}$ $+\frac{\int_0^1 t^{k+\alpha}\ (1-qt)_q^{n-k+\alpha} f(t)d_qt}{B_q(k+\alpha+1,n-k+\alpha+1)} - \frac{\int_0^1 t^{k-1}\ (1-qt)_q^{n-k-1} f(t)d_qt}{B_q(k,n-k)} = F_1 + F_2 + F_3$.

We consider $I_k = \frac{\int_0^1 t^{k+\alpha}\ (1-qt)_q^{n-k+\alpha}\ \left(f(q^{\alpha+1}t)-\widetilde{f}_k(t)\right)d_qt}{B_q(k+\alpha+1,n-k+\alpha+1)}$, with $\widetilde{f}_0(t) = f(0)$, $\widetilde{f}_n(t)$ $= f(1)$ and $\widetilde{f}_k(t) = f(t)$, $k = 1,\dots,n-1$ and prove that $\lim_{\alpha\to-1} I_k = 0$, for $k = 0,\dots,n$. We use the additivity of the modulus of continuity of f, Beta integrals and we set $\delta = [\alpha+1]\,/\,[n+2\alpha+2]$. We have $|f(q^{\alpha+1}t) - f(0)| \le \omega(f,t) \le \omega(f,\delta)(1+t/\delta)$, hence $|I_0| \le \omega(f,\delta)(1+\frac{1}{\delta}\int_0^1 t^{\alpha+1}\ (1-qt)_q^{n+\alpha}d_qt \Big/ \int_0^1 t^{\alpha}\ (1-qt)_q^{n+\alpha}\ d_qt) \le 2\omega(f,\frac{[\alpha+1]}{[n+2\alpha+2]})$.

For $k = n$, we have $|f(q^{\alpha+1}t) - f(1)| \le \omega(f, 1-q^{\alpha+1}t)$, hence

$|I_n| \le \omega(f,\delta)(1+\int_0^1 t^{\alpha+n}\ (1-qt)_q^{\alpha+1}d_qt \Big/ (\delta\int_0^1 t^{\alpha+n}\ (1-qt)_q^{\alpha}\ d_qt)) \le 2\omega(f,\frac{[\alpha+1]}{[n+2\alpha+2]})$.

For $k = 1,\dots,n-1$, we have $|f(q^{\alpha+1}t) - f(t)| \le \omega(f, 1-q^{\alpha+1})$ and

$|I_k| \le \omega(f, 1-q^{\alpha+1})$. As $f_{n,0,q}^{\alpha,\alpha} - f(0) = I_0$ and $f_{n,n,q}^{\alpha,\beta} - f(1) = I_n$, the result follows for $k = 0$ and n.

For the other cases, $F_1 = I_k$ vanishes when α tends to -1. The upper term of F_2 is the q-integral $(1-q)\sum_{j=0}^n q^{j(k+\alpha+1)}(1-q^{j+1})^{n-k+\alpha} f(q^j)$. This serie is uniformly convergent, hence its limit when α tends to -1 is the upper term of F_3. At last the lower term of F_2 tends to the lower term of A_3 because Γ_q is continuous. ∎

Numerous properties shown in the case $\alpha,\beta > -1$ are still true if $\alpha = \beta = -1$. Some of them are given in the following.

Proposition 7 *If the function f is continuous at the points* 0 *and* 1, *and verifies the*

condition C(-1), we have :

$$D_q M^{-1,-1}_{n,q} f(x) = M^{0,0}_{n-1,q}\left(D_q f\left(\frac{\cdot}{q}\right)\right)(qx), x \in [0,1]. \tag{13}$$

Proof. The expressions for $(f^{\alpha,\beta}_{n,k+1} - f^{\alpha,\beta}_{n,k})$ in proposition (1) hold if $\alpha = \beta = -1$ and $k = 1, \ldots, n-2$. For the two other terms we have

$[n-1]\,(f^{-1,-1}_{n,1} - f(0)) = -\left[t^{n-1}(f(t) - f(1))\right]_0^1 + \int_0^1 (1-qt)_q^{n-1} D_q f(t) d_q t$ and

$[n-1]\,(f^{-1,-1}_{n,n-1} - f(1)) = -\left[(1-t)_q^{n-1}(f(t) - f(0))\right]_0^1 + \int_0^1 (qt)^{n-1} D_q f(t) d_q t.$

The first terms vanish since f is continuous at 0 and 1. ∎

Theorem 6

1. *$M^{-1,-1}_{n,q} f(x) = \sum_{j=0}^{\infty} \Phi^{-1,-1}_{j,n,q}(x) f(q^j)$, where $\Phi^{-1,-1}_{j,n,q}$ is defined in formula (6) with $\alpha = \beta = -1$. The sequence $\Phi^{-1,-1}_{r,n,q}, \Phi^{-1,-1}_{r-1,n,q}, \ldots, \Phi^{-1,-1}_{0,n,q}$ is totally positive. Consequently the operator $M^{-1,-1}_{n,q}$ diminishes the number of sign changes and preserves the monotony.*

2. *If f is continuous on $[0,1]$, then $\left\|M^{-1,-1}_{n,q} f - f\right\|_\infty \leq Cte\ \omega\left(f, \frac{1}{\sqrt{[n]_q}}\right)$, (theorem 2).*

3. *If $\lim_{n\to\infty} q_n = 1$ and if the function f is bounded on $[0,1]$, then*

   (a) *$\lim_{n\to\infty} M^{-1,-1}_{n,q_n} f(x) = f(x)$ if f is continuous at the point $x \in]0,1[$, (theorem 3).*

   (b) *$\lim_{n\to\infty} [n]_{q_n} (M^{-1,-1}_{n,q_n} f(x) - f(x)) = f''(x)$ if the function f admits a second derivative at the point $x \in]0,1[$, (theorem 4).*

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